RADICALS

Definition: A **radical** is said to be in **simplest form** if the following conditions are satisfied:

1. The radicand has no factor with an exponent greater than or equal to the root index (note that this may require you to prime-factor the coefficient).
2. The radicand is written with a positive coefficient.
3. The radicand does not contain a fraction.
4. The root index is reduced to the smallest possible value.
5. The denominator of a fraction does not contain a radical expression.

To simplify radicals, we use the following techniques:

I. Removing Factors

Use the property that $\sqrt[n]{ab} = \sqrt[n]{a}\,\sqrt[n]{b}$ and the property that

$$\sqrt[n]{a^n} = \begin{cases} a & \text{if } a \geq 0 \text{ or } n \text{ is odd} \\ |a| & \text{if } n \text{ is even} \end{cases}$$

It is also useful to prime-factor the numerical coefficient. If the numerical coefficient is negative (and n is odd), remove the factor of -1.

Example 1

(a) $\sqrt{25x^5y^4} = \sqrt{25x^4y^4}\,\sqrt{x} = 5x^2y^2\,\sqrt{x}$

(b) $\sqrt[3]{-80x^4y^5z^6} = \sqrt[3]{-1\cdot 2^4\cdot 5\;x^4y^5z^6} = \sqrt[3]{-1\cdot 2^3\;x^3y^3z^6}\;\sqrt[3]{2\cdot 5\;xy^2}$

$= -2xyz^2\,\sqrt[3]{10xy^2}$

II. Rationalizing the Denominator

If there is a fraction under the radical, use the property that

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$$ to separate the expression into two radicals.

To eliminate the radical in the denominator, rename the fraction by multiplying the numerator and denominator by the smallest nth root possible that will turn the expression in the denominator into a perfect nth power.

If the denominator of a fraction contains a radical expression with two terms, it is necessary to multiply the numerator and denominator by the **conjugate** of the denominator to rationalize the denominator (see example 5(e)).

<u>Example 2</u>

(a) $$\sqrt{\frac{x}{4y}} = \frac{\sqrt{x}}{\sqrt{4y}} = \frac{\sqrt{x}}{\sqrt{4y}} \cdot \frac{\sqrt{y}}{\sqrt{y}} = \frac{\sqrt{xy}}{\sqrt{4y^2}} = \frac{\sqrt{xy}}{2y}$$

(b) $$\frac{\sqrt[3]{x}}{\sqrt[3]{4y}} = \frac{\sqrt[3]{x}}{\sqrt[3]{2^2y}} = \frac{\sqrt[3]{x}}{\sqrt[3]{2^2y}} \cdot \frac{\sqrt[3]{2y^2}}{\sqrt[3]{2y^2}} = \frac{\sqrt[3]{2xy^2}}{\sqrt[3]{2^3y^3}} = \frac{\sqrt[3]{2xy^2}}{2y}$$

III. Reducing the Index

If the root index and all the exponents of the factors under the radical have a common factor, this factor can be divided out, thereby reducing the index. Be sure to prime-factor the numerical coefficient before applying this technique.

<u>Example 3</u>

(a) $$\sqrt[10]{y^6} = \sqrt[5]{y^3}$$

(b) $$\sqrt[6]{8x^9y^6} = \sqrt[6]{2^3x^9y^6} = \sqrt[2]{2x^3y^2}$$

IV. Adding and Subtracting Radical Terms

To add or subtract radical terms, the terms must be "like" terms. Sometimes, it is necessary to first simplify the radical terms before it becomes clear that they are like terms.

Example 4

(a) $\sqrt{32} + 5\sqrt{18} = \sqrt{16 \cdot 2} + 5 \cdot \sqrt{9 \cdot 2} = 4\sqrt{2} + 5 \cdot 3\sqrt{2}$
$= 4\sqrt{2} + 15\sqrt{2} = 19\sqrt{2}$

(b) $\sqrt[3]{54x^4y} - x\sqrt[3]{16xy} = \sqrt[3]{2 \cdot 3^3x^4y} - x\sqrt[3]{2^4xy}$
$= \sqrt[3]{3^3x^3}\sqrt[3]{2xy} - x\sqrt[3]{2^3}\sqrt[3]{2xy}$
$= 3x\sqrt[3]{2xy} - 2x\sqrt[3]{2xy} = x\sqrt[3]{2xy}$

V. Multiplying Radical Factors

To multiply radical factors with the same index, we multiply their radicands and then simplify using the above techniques if possible. When multiplying expressions containing radical terms, we can also use the distributive property and multiplication formulas that were previously used to square binomials and multiply binomials.

Example 5

(a) $\sqrt{14x} \cdot \sqrt{7xy} = \sqrt{2 \cdot 7^2x^2y} = \sqrt{7^2x^2} \cdot \sqrt{2y} = 7x\sqrt{2y}$

(b) $4\sqrt[3]{2x^2} \cdot 3\sqrt[3]{12xy^2} = 12\sqrt[3]{24x^3y^2} = 12\sqrt[3]{2^3x^3}\sqrt[3]{3y^2}$
$= 12 \cdot 2x\sqrt[3]{3y^2} = 24x\sqrt[3]{3y^2}$

(c) $2\sqrt{3}(5\sqrt{3} + 4\sqrt{2}) = 2\sqrt{3} \cdot 5\sqrt{3} + 2\sqrt{3} \cdot 4\sqrt{2} = 10 \cdot 3 + 8\sqrt{6}$
$= 30 + 8\sqrt{6}$

(d) $(2\sqrt{x} - 3\sqrt{y})^2 = (2\sqrt{x})^2 - 2(2\sqrt{x})(3\sqrt{y}) + (3\sqrt{y})^2$
$= 4x - 12\sqrt{xy} + 9y$

(e) $\dfrac{2 + \sqrt{3}}{4\sqrt{3} - \sqrt{2}} = \dfrac{2 + \sqrt{3}}{4\sqrt{3} - \sqrt{2}} \cdot \dfrac{4\sqrt{3} + \sqrt{2}}{4\sqrt{3} + \sqrt{2}} = \dfrac{(2 + \sqrt{3})(4\sqrt{3} + \sqrt{2})}{(4\sqrt{3} - \sqrt{2})(4\sqrt{3} + \sqrt{2})}$

$= \dfrac{8\sqrt{3} + 2\sqrt{2} + 4 \cdot 3 + \sqrt{6}}{16 \cdot 3 + 4\sqrt{6} - 4\sqrt{6} - 2} = \dfrac{8\sqrt{3} + 2\sqrt{2} + 12 + \sqrt{6}}{46}$

Math 20

Radicals - Homework Assignment

Simplify each of the following radicals:

1. $\sqrt[4]{9x^6y^7}$

2. $\sqrt[5]{16x^3y^5}$

3. $\sqrt[8]{3^4}$

4. $\sqrt[9]{y^6}$

5. $\sqrt[4]{x^2}$

6. $\sqrt[6]{y^4(x+y)^2}$

7. $\sqrt[6]{81y^8}$

8. $\sqrt[4]{4x^{10}}$

9. $\sqrt{\frac{1}{2}}$

10. $\sqrt[3]{\frac{1}{2}}$

11. $\sqrt[3]{-\frac{3}{4}}$

12. $\sqrt[4]{\frac{5}{8}}$

13. $\sqrt{\frac{9}{x}}$

14. $\sqrt[3]{\frac{x}{y}}$

15. $\sqrt[4]{\frac{2x}{y^5}}$

16. $\sqrt{\frac{5}{4}-\frac{4}{5}}$

Perform the following operations and simplify your answers:

17. $3\sqrt{72} - 5\sqrt{50}$

18. $3\sqrt{28} - 10\sqrt{63}$

19. $4\sqrt{\frac{1}{5}} + 2\sqrt{125}$

20. $\sqrt{56} - \frac{1}{2}\sqrt{\frac{2}{7}}$

21. $2\sqrt[3]{-54} - \sqrt[3]{128}$

22. $3\sqrt[4]{32} - 5\sqrt[4]{2}$

23. $\sqrt{8x} + \sqrt{72x}$

24. $\sqrt{20xy^2} - 2\sqrt{45x^3}$

25. $2\sqrt[3]{54x^4y} - \sqrt[3]{\frac{xy^4}{4}}$

26. $\sqrt[4]{\frac{x}{y}} - \sqrt[4]{xy^3}$

27. $\sqrt{4 + 4x} + \sqrt{16 + 16x}$

28. $\sqrt{6} \cdot \sqrt{2}$

29. $\sqrt[3]{6} \cdot \sqrt[3]{2}$

30. $\sqrt[4]{8} \cdot \sqrt[4]{2}$

31. $\sqrt{15x} \cdot \sqrt{6x}$

32. $-3\sqrt{8xy} \cdot 4\sqrt{3x^3y^2}$

33. $\sqrt[3]{4x^2y} \cdot \sqrt[3]{6x^2y^3}$

Perform the following operations and simplify your answers:

34. $4\sqrt[4]{27x^3} \cdot 5\sqrt[4]{3x^5}$

35. $2\sqrt{5}\,(4\sqrt{10} - 3\sqrt{5})$

36. $(3 + \sqrt{5})(3 - \sqrt{5})$

37. $(2\sqrt{3} - 4\sqrt{5})(2\sqrt{3} + 4\sqrt{5})$

38. $(1 + \sqrt{2})^2$

39. $(2 - \sqrt{5})^2$

40. $(3 - \sqrt{2})(5 - \sqrt{6})$

41. $(3\sqrt{3} - 2\sqrt{6})(2\sqrt{6} - 3\sqrt{3})$

42. $(\sqrt{x} - \sqrt{y})^2$

43. $(\sqrt{x + y})^2$

44. $\dfrac{\sqrt{24}}{\sqrt{3}}$

45. $\dfrac{\sqrt{10}}{\sqrt{3}}$

46. $\dfrac{\sqrt{18}}{\sqrt{3x}}$

47. $\dfrac{4\sqrt{5x}}{\sqrt{2x^3}}$

48. $\dfrac{\sqrt[3]{-3y}}{\sqrt[3]{48y}}$

49. $\dfrac{\sqrt[4]{7x^2}}{\sqrt[4]{2x}}$

50. $\dfrac{\sqrt[5]{5y^4}}{\sqrt[5]{3y^2}}$

51. $\dfrac{-4}{3\sqrt{12}}$

52. $\dfrac{1}{\sqrt[3]{2}}$

53. $\dfrac{3}{1 + \sqrt{2}}$

54. $\dfrac{5 + 2\sqrt{7}}{2 - 2\sqrt{7}}$

Perform the following operations and simplify your answers:

55. $\dfrac{\sqrt{x}}{\sqrt{x} + \sqrt{y}}$

56. $\dfrac{a^2}{\sqrt{a^2 + b^2}} + \dfrac{b^2}{\sqrt{a^2 + b^2}}$

57. $\sqrt{R^2 - x^2} + (R + x)\dfrac{-x}{\sqrt{R^2 - x^2}}$

58. $\dfrac{\dfrac{x^2}{\sqrt{x^2 + 1}} - \sqrt{x^2 + 1}}{x^2}$

59. $-\dfrac{1}{2}\sqrt{\dfrac{x}{y}} \cdot \dfrac{x\sqrt{\dfrac{y}{x}} - y}{x^2}$

60. If $f(x) = x^2 - 2x + 5$, find $f(1 + \sqrt{2})$

61. Find the area of the right triangle with vertices at (1,1), (-3,2), and (-4,-2).