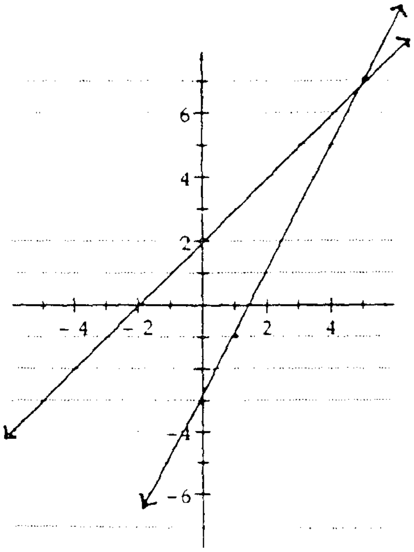
Solving Systems of Linear Equations

I. GRAPHICAL METHOD

When solving a system with two linear equations in two variables, we are looking for the point where the two lines cross. This can be determined by graphing each line on the same coordinate system and estimating the point of intersection. Sometimes, the lines do not cross in which case they are parallel. A system consisting of two **parallel lines** is said to be **inconsistent** and has **no solutions**. Other times, the **two lines coincide** and any point on the line will be a solution to the system. This type of system is said to be **dependent** and has an **infinite number of solutions**. When two lines cross in **exactly one point**, the system is **consistent and independent** and the solution is the one ordered pair where the two lines cross. The coordinates of this ordered pair can be estimated from the graph of the two lines. These three cases are illustrated below. The graphical method is good because it clearly illustrates the principle involved. However, it takes a lot of time, does not always give us an **exact** solution, and cannot be used when we have more than two variables in the equations. When we want exact solutions or want to solve systems with more than two equations in two variables, we must use algebraic methods described in the next section.

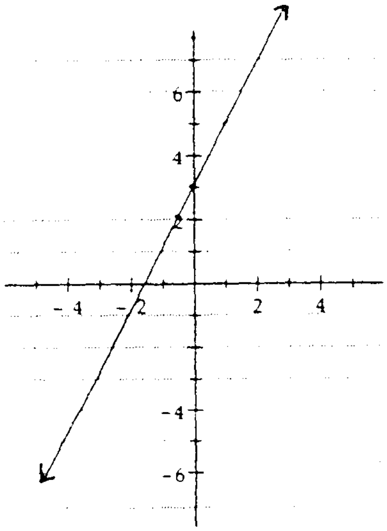
Case 1: Consistent & Independent System

$y = x + 2$
$2x - y = 3$



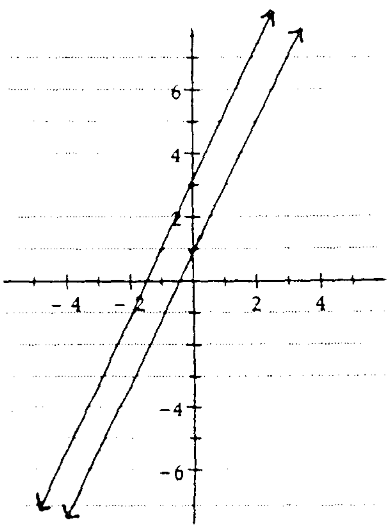
Case 2: Dependent System

$y = 2x + 3$
$2x - y = -3$



Case 3: Inconsistent System

$y = 2x + 1$
$2x - y = -3$



II. ALGEBRAIC METHODS

Recall from algebra that there are two basic algebraic methods of solving linear systems of equations: the **substitution method** and the **elimination** or **addition method**. The goal in each case is to end up with **one equation** in **one variable.**

To solve the system illustrated in case 1 above using the **substitution method,** we would replace the y in the second equation with the expression x + 2 from the first equation: $2x - (x + 2) = 3$. This is now easy to solve for x:

$$2x - x - 2 = 3$$
$$x - 2 = 3$$
$$x = 5$$

Thus, the x-coordinate of the solution is 5. To find the y-coordinate, we substitute 5 in for x in either of the original equations. It is easier to use the first equation:

$$y = 5 + 2$$
$$y = 7$$

So, the solution to the system is the ordered pair (5,7). This is the point where both lines cross as we saw in section I.

To solve the system illustrated in case 2 above using the **substitution method,** we would replace the y in the second equation with the expression 2x + 3 from the first equation: $2x - (2x + 3) = -3$

$$2x - 2x - 3 = -3$$
$$-3 = -3$$

This gives us a true statement which means that the system is <u>Dependent</u>. There are an **<u>infinite number of solutions</u>**.

To solve the system illustrated in case 3 above using the **substitution method,** we would replace the y in the second equation with the expression 2x + 1 from the first equation: $2x - (2x + 1) = -3$

$$2x - 2x - 1 = -3$$
$$-1 = -3$$

This gives us a false statement which means that the system is <u>Inconsistent</u>. There are **<u>no solutions</u>**.

Many systems are easier to solve using the **elimination or addition method**. This is especially true when we have a system with more than two variables. We can also avoid fractions when using this method. In the elimination/addition method, we multiply one or both of the equations by a constant so that when we add them together, we eliminate one of the variables. Remember that multiplying an equation by a constant produces an equivalent equation. The goal is to eventually end up with one equation in one variable.

Example: Solve the following system using the **elimination/addition method**:

$$\begin{cases} 3x - 2y = 27 & (1) \\ 2x + 5y = -1 & (2) \end{cases}$$

To eliminate y, we could multiply the first equation by 5 and the second equation by 2, and add them together:

$$\begin{array}{lrl} & 5\,(1): & 15x - 10y = 135 \\ + & \underline{2\,(2):} & \underline{\;\;4x + 10y = -2} \\ & & 19x \qquad\;\; = 133 \\ & & \quad x = 7 \end{array}$$

To find the y-coordinate, we substitute in 7 for x in either of the original two equations:

$$\begin{aligned} 3(7) - 2y &= 27 \\ 21 - 2y &= 27 \\ -2y &= 6 \\ y &= -3 \end{aligned}$$

Thus the solution is $(7, -3)$.

Example: Solve the following system using the **elimination/addition method**:

$$\begin{cases} 2x - 4y + 3z = 31 & (1) \\ 5x - 2y - 2z = 6 & (2) \\ 3x + 4y + 5z = 19 & (3) \end{cases}$$

We must choose a variable to eliminate, and eliminate it using two different pairs of equations. It appears that y is the easiest variable to eliminate.

$$\begin{array}{lrl} & (1): & 2x - 4y + 3z = 31 \\ + & \underline{-2\,(2):} & \underline{-10x + 4y + 4z = -12} \\ & (4): & -8x \qquad + 7z = 19 \end{array} \qquad \begin{array}{lrl} & (1): & 2x - 4y + 3z = 31 \\ + & \underline{(3):} & \underline{3x + 4y + 5z = 19} \\ & (5): & 5x \qquad + 8z = 50 \end{array}$$

Equations (4) and (5) now form a reduced system with only two equations in two variables. We solve it in a similar manner by eliminating x:

$$\begin{array}{lrl} & 5\,(4): & -40x + 35z = 95 \\ + & \underline{8\,(5):} & \underline{\;\;40x + 64z = 400} \\ & & \qquad 99z = 495 \\ & & \qquad\;\; z = 5 \end{array}$$

We go back to our reduced system to now find x using equation (5):

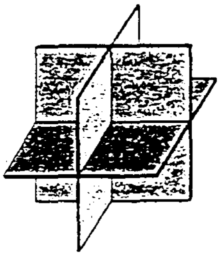
$$\begin{aligned} 5x + 8(5) &= 50 \\ 5x + 40 &= 50 \\ 5x &= 10 \\ x &= 2 \end{aligned}$$

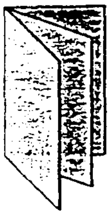
We can find the remaining variable y by substituting both $z = 5$ and $x = 2$ into one of the original equations:

$$\begin{aligned} (1): \quad 2(2) - 4y + 3(5) &= 31 \\ 4 - 4y + 15 &= 31 \\ -4y + 19 &= 31 \\ -4y &= 12 \\ y &= -3 \end{aligned}$$

Thus the ordered triple $(x,y,z) = (2,-3,5)$ is the solution to the system.

Note that the solution to the last example was an ordered triple which is a point in space using a three-dimensional Cartesian coordinate system. In this coordinate system, the graph of a linear equation in three variables is a plane. The fact that we obtained just one ordered triple as a solution to the last example indicates that the three planes intersected in a single point. (See figure (a) below.) There are many other possibilities, as illustrated below. In figures (b), (c), and (d) the intersection is either a line or an entire plane, so the corresponding system has infinitely many solutions. As before, a system like this is called **dependent.** In figures (e), (f), (g), and (h) the three planes have no common intersection, so the corresponding system has no solution. A system like this is called **inconsistent**. Algebraically, we recognize a dependent system if at any step in the process we eliminate all the variables but end up with a true statement (like 0 = 0). If at any step in the process we eliminate all the variables but end up with a false statement (like 0 = 7) then the system is **inconsistent.**


(a)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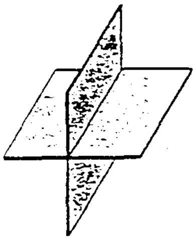
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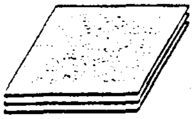
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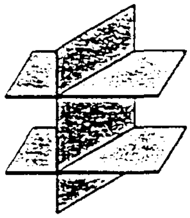
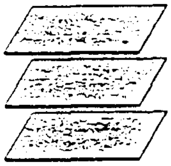
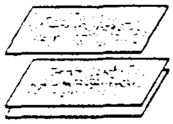
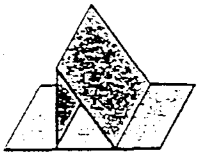

(e)


(f)


(g)


(h)

Example: Solve the following system using the **elimination/addition method**:

$$\begin{cases} 2x + 3y - z = 10 & (1) \\ 3x - 2y + z = 7 & (2) \\ -4x - 6y + 2z = 9 & (3) \end{cases}$$

Eliminate z using two different pairs of equations:

$$\begin{array}{lll} & (1): & 2x + 3y - z = 10 \\ + & \underline{(2):} & \underline{3x - 2y + z = 7} \\ & (4): & 5x + y \quad = 17 \end{array} \qquad \begin{array}{ll} -2\,(2): & -6x + 4y - 2z = -14 \\ \underline{(3):} & \underline{-4x - 6y + 2z = \quad 9} \\ (5): & -10x - 2y \quad = -5 \end{array}$$

Eliminate y using equations (4) and (5) by first multiplying equation (4) by 2 and then adding it to equation (5):

$$\begin{array}{ll} 2\,(4): & 10x + 2y = 34 \\ \underline{(5):} & \underline{-10x - 2y = -5} \\ & \quad 0 \quad = 29 \end{array}$$

Since this is a false statement, the system is **Inconsistent** and has **no solution.**

SYSTEMS OF EQUATIONS

Find all solutions using either substitution or elimination. If there is not a solution or if there are infinitely many solutions, state so.

1. $\begin{aligned} 2x+3y&=12 \\ y&=2x-4 \end{aligned}$

2. $\begin{aligned} 2x+y&=3 \\ 3x+5y&=1 \end{aligned}$

3. $\begin{aligned} 2x-2y&=0 \\ x&=y-1 \end{aligned}$

4. $\begin{aligned} \frac{2}{3}x-y&=0 \\ 10x+4y&=19 \end{aligned}$

5. $\begin{aligned} \frac{4}{3}x+\frac{1}{5}y&=3 \\ \frac{2}{3}x-\frac{3}{5}y&=5 \end{aligned}$

6. $\begin{aligned} -2x+6y&=3 \\ 4x-12y&=-6 \end{aligned}$

7. $\begin{aligned} 3x-5y&=-13 \\ 4x+2y&=0 \end{aligned}$

8. $\begin{aligned} 1.2x+2.5y&=4 \\ 0.8x-1.5y&=-10 \end{aligned}$

9. $\begin{aligned} \frac{x-3}{2}&=\frac{y-5}{4} \\ \frac{x+5}{2}&=\frac{2y+7}{5} \end{aligned}$

10. $\begin{aligned} 5x+2y&=0 \\ -3z&=12 \\ 6y+5z&=10 \end{aligned}$

11. $\begin{aligned} x+y&=5 \\ 3x+z&=2 \\ 4y-z&=8 \end{aligned}$

12. $\begin{aligned} 2x+2z&=2 \\ 3y-4z&=4 \\ 5x+3y&=4 \end{aligned}$

13. $\begin{aligned} x+2y+6z&=5 \\ -x+y-2z&=3 \\ x-4y-2z&=1 \end{aligned}$

14. $\begin{aligned} x-y+3z&=-8 \\ 2y-z&=15 \\ 3x+2z&=-7 \end{aligned}$

15. $\begin{aligned} x+4y-2z&=2 \\ x+y+z&=-1 \\ 5x+7y+3z&=-3 \end{aligned}$

16. $\begin{aligned} 2x+y-2z&=4 \\ 3x-2y+4z&=6 \\ -4x+y+6z&=12 \end{aligned}$

Solving Systems of Linear Equations Using Matrices

I. Gauss–Jordan Method

A very systematic method of solving linear systems of equations is called the Gauss-Jordan Method. This method involves the use of matrices, the plural of the word matrix. A **matrix** is a rectangular array of numbers arranged in rows and columns. The numbers in the array are called the elements of the matrix.

To solve linear systems of equations, we will first form the **augmented matrix** by writing the numerical coefficients and constants of each equation in a matrix. Before forming the matrix, be sure to line-up the variables so that they are in the same column; any missing terms will require a coefficient of 0. We separate the coefficients from the constants with a vertical line. Two examples are:

System of Linear Equations — **Augmented Matrix**

$$\begin{cases} 5x + 8y = 2 \\ 7x - 3y = 88 \end{cases} \qquad \left[\begin{array}{cc|c} 5 & 8 & 2 \\ 7 & -3 & 88 \end{array}\right]$$

$$\begin{cases} x + 2y + 4z = 24 \\ 2x - y - 3z = -22 \\ 3x \phantom{{}-y} + 5z = 19 \end{cases} \qquad \left[\begin{array}{ccc|c} 1 & 2 & 4 & 24 \\ 2 & -1 & -3 & -22 \\ 3 & 0 & 5 & 19 \end{array}\right]$$

If two systems of linear equations have the same solution sets, then the systems are said to be equivalent. Similarly, the augmented matrices of equivalent systems are also equivalent.

Example: Verify that the augmented matrices A and B are equivalent:

$$A = \left[\begin{array}{ccc|c} 1 & 6 & -2 & -14 \\ 3 & 0 & 2 & 4 \\ 5 & -3 & 3 & 1 \end{array}\right] \qquad B = \left[\begin{array}{ccc|c} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 8 \end{array}\right]$$

By definition, the two augmented matrices are equivalent if the associated system of linear equations have the same solution sets. The solution set of the system associated with matrix B can be readily identified by converting back to equations:

$$\begin{cases} x = -4 \\ y = 1 \\ z = 8 \end{cases}$$